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DIOPHANTINE EQUATIONS OF THE TYPE: $(a^m + b^m + c^m = d^n)$ & $(a^m + b^m + c^m + d^m = e^n)$

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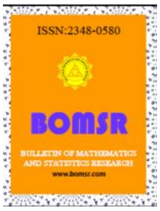

ABSTRACT

In this paper the authors have considered, Diophantine equations, $(a^m + b^m + c^m = d^n)$ & $(a^m + b^m + c^m + d^m = e^n)$ for $m = (2,3,4)$ & $n < 10$. While the authors [ref. (2)] in their year 2023 paper discussed above equation for n=2 & 'm' for various degrees, this paper has done a systematic approach for degree n=2 & higher. In the previous paper the authors had considered three terms on the (LHS), in this paper we have also considered four terms on the (LHS). For degree six only numerical solution is given since a parametric solutions has been evasive.

We have considered equations of the below type:

$$(a^p + b^p + c^p) = (d)^q \quad \& \quad (a^p + b^p + c^p + d^p) = (e^q)$$

Above can be written as:

$$. (a,b,c)^p = (d)^q \ \& \ (a,b,c,d)^p = (e)^q$$

Section 1: $(a^p + b^p + c^p) = (d)^q$)

[$Three\ terms\ on\ (LHS)$]

$(a, b, c)^p = (d)^q$

$(p, q) = (2,2)$

We take:

$(a, b, c) = [(3p), (3q), (pq)]$

Where, $(p, q) = (3k-2), (3k-5)$ & $d = (9k^2 - 21k + 19)$

For $k = 0$

$(15,10,6)^2 = (19)^2$

For $k = 3$

$(21,12,28)^2 = (37)^2$

2. $\boldsymbol{(a, b, c)^p = (d)^q}$

$(a, b, c)^3 = (d)^2$

$(p, q) = (3,2)$

We take $(a, b, c) = [(x-1), x, (x+1)]$ & *we get*:

$d^2 = 3x(x^2 + 2)$ & *we take*

$x = 6m^2$ & *we get*:

$d^2 = (6m)^2(18m^4 + 1)$,

For $m = 2$, *we get* $d = 204$

$(a, b, c) = (23,24,25)$ & $d = (204)$

***Another numerical solution for*: $\boldsymbol{(p, q) = (3,2)}$**

$(3,2,1)^3 = (6)^2$

We take:

$(a, b, c) = (3 + kt, 2 + t, 1)$

$d = (6 + kt + t)$. *For*, $t = \left[\frac{-3k}{k^2-k+1}\right]$

we get: $(a, b, c) = (-6,5,7)$ & $d = 6$

***Another numerical solution is*:**

For, $(p, q) = (3,2)$

For n=3, we have:

$(49,35,126)^3 = (1470)^2$

$(a, b, c)^p = (d)^q$

3. $(p, q) = (4,2)$

we take: $(a, b, c) = [(uv), (uw), (vw)]$ & $d = [v^4 + (vw)^2 + w^4]$

Where, $u^2 = v^2 + w^2$

For $(u, v, w) = (5,4,3)$ *we get*: $(15,12,20)^4 = (481)^2$

$Also, (a, b, c)^4 = (a^2 - b^2 + c^2)^2$

$(a, b, c)^p = (d)^q$

4. $(p, q) = (5,2)$

We have numerical solution:

$Where, (a, b, c, d) = (48, -30, -18, 15120)$

5. $(a, b, c)^p = (d)^q$

$(p, q) = (6,2)$

We have numerical solution: $(42,81,100)^6 = (1134865)^2$

Section -2

[*Four terms on* (LHS)]

$(a^p + b^p + c^p + d^p) = (e^q)$

Above can be written as: $(a, b, c, d)^p = (e)^q$

1. $(p, q) = (2,2)$

We take: $a = n(k + 4), b = 2m(k - 5), c = 2mn, d = 2mn \,\&\, e = (29k^2 - 2k + 14)$

$Where, (m, n) = [(2k - 1), (5k + 2)]$

For $k = 0$ *we get*: $(8,10,4,4)^2 = (14)^2$

Alternate Identity is given below:

We have numerical solution shown below: $(2,4,7,10)^2 = (13)^2$

We take $(a + b + c) = (e)$

We get $d^2 = 2(ab + bc + ca)$

Parameterising at:

$(a, b, c, d) = [(2 + t), (4 + t), 7 + t), (10 + kt), (13 + 3t)]$

$a = 2(k^2 - 10k + 20)$

$b = 2(2k^2 - 10k + 14)$

$c = 2(5k^2 - 26k + 30)$

$d = (7k^2 - 20k + 10)$

$e = (13k^2 - 60k + 78)$

For, $k = 0$ *we get*: $(20,14,5,30)^2 = (39)^2$

$(a, b, c, d)^p = (e)^q$ ..... (1)

2. $(p, q) = (3,3)$

$(18,17, -20, -14)^3 = (a + b + c + d)^3 = (1)^3$ ..... (2)

$(a, b, c, d) = (18,17, -20, -14)$

From the above, we have the relationship, $(a + b + c + d) = 1 = (e)$

Also from (1) *we get*:

From the numerical equality (2) *above we have the relationship*:

$7c = -4(a+b)$ ….. $(3a)$

$49ab = (4a+4b+7)(4-7d)$ ….. $(3b)$

Using the above $(3a$ & $3b)$*relationship and*

substituting in equation (1) *above* & *also using*

maple math software we get the condition:

$12(a-b)^2 + 9(a+b) = ab + 21$ ….. (4)

Parametrizing equation (4) *above at* $(a,b) = (18,17)$

we get another numerical value: $(a,b) = (\frac{50}{3}, 17)$

Substituting this value in $(3a)$ & $(3b)$

we get: $(c,d) = (-\frac{404}{21}, -\frac{282}{21})$

Hence, (*new*) *numerical solution is*:

$(350,357,-404,-282)^3 = (a+b+c+d)^3 = (21)^3$

$(a,b,c,d)^p = (e)^q$ − − − − − − − − (1) $(p,q) = (3,3)$

$(18,17,-20,-14)^3 = (a+b+c+d)^3 = (1)^3$ ……….(2)

Alternate method inorder to parametrize above equation (1) is shown below:

$(a^3 + b^3 + c^3 + d^3) = (a+b+c+d)^3$

Hence,

$(a,b,c,d)^3 = (a,b,c,d)^2 + 2ab(c+d) + 2cd(a+b)$

$(a,b,c,d) = (18,17,-20,-14)$

$1 = 1 + 2ab(c+d) + 2cd(a+b)$.

Hence we have:

$-(c+d)cd = (a+b)(ab+c+d)$

Parametrizing at, $(a+b) = 35$ & $(c+d) = -34$

We have,

$(a,b,c,d) = (18+t, 17-t, -20+kt, -14-kt)$ &

we get: $t = \frac{35+204k}{34k^2-35}$

Hence a parametric form is shown below:

$(612x^2 + 204xy - 595y^2), (578x^2 - 204xy - 630y^2),$

$(a,b,c,d,e) = [(-476x^2 + 35xy + 700y^2),] (-680x^2 - 35xy + 490y^2), (34x^2 - 35y^2).$

For $k = 1$

we get $(a,b,c,d,e) = (256,-221,225,-259,1)$ *as new solution*.

For $k = 2$

we get (a,b,c,d)= $(2261,1274,-1134,-2300,101)$ *as new solution*

$(a,b,c,d)^p = (e)^q$

3. $(p,q) = (3,4)$,

we have numerical solution: $(7,6,2,9)^3 = (6)^4$

The above has the relationship: $(a + b + c - d = e)$

Another numerical solution is:

Also, $(18,17,-20,-14)^3 = (1)^4 = e^4$

Above has the relationship: $(a + b + c + d = e)$

$(a,b,c,d)^p = (e)^q$

4. $(p,q) = (4,2)$

$(a,b,c,d) = (p^2, pq, q^2, pq)$ & $e = (p^4 + q^4)$

5. $X^4 + Y^4 + Z^4 + W^4 = T^3$

$(a,b,c,d)^p = (e)^q$

$(p,q) = (4,3)$

We prove that there are many parametric solutions of $X^4 + Y^4 + Z^4 + W^4 = T^3$

$$X^4 + Y^4 + Z^4 + W^4 = T^3$$

Taking X=a, Y=b, Z=a+b, then we get:

$$2(a^2 + ab + b^2)^2 + W^4 = T^3$$

If we find the rational solution of equation (2), we can obtain the parametric solution of (1).

Some solutions were found where |a,b| < 100 and W < 1000 (see table below):

(A).	(18,9,99,459),(27,−18,99,459),(27,−9,99,459),(51,42,135,747),(62,36,1 15,657),
(B).	(68,−34,17,289),(75,63,51,747),(78,14,115,657),(92,−78,115,657),(92,− 14,115,657),
(C)	(93,−51,135,747),(93,−42,135,747),(93,42,51,747),(98,−62,115,657), (98,−36,115,657).

It seems that equation (1) has infinitely many parametric solutions.

We show two examples for (a,b,W,T) = (18,9,99,459) and (51,42,135,747) (a,b,W,T) = (18,9,99,459)

$$X = -27m^6 - 54nm^5 + 270n^2m^4 + 540n^3m^3 + 135n^4m^2 - 108n^5m - 27n^6,$$
$$Y = 9m^6 - 108nm^5 - 405n^2m^4 - 180n^3m^3 + 270n^4m^2 + 162n^5m + 9n^6,$$
$$Z = -18m^6 - 162nm^5 - 135n^2m^4 + 360n^3m^3 + 405n^4m^2 + 54n^5m - 18n^6,$$
$$W = 99(m^2 + nm + n^2)^3,$$
$$T = 459(n^2 + nm + m^2)^4$$

6. $X^4 + Y^4 + Z^4 + W^4 = T^n$

$(a,b,c,d)^p = (e)^q$

$(p,q) = (4,n)$

Taking X=a, Y=b, Z=a+b, then we get:

$$2(a^2 + ab + b^2)^2 + W^4 = T^n$$

Consider (*a,b,W,T*) and let

(a_0, b_0, W_0, T_0) (be a known solution).

Let (*a,b*) is a solution of $a^4 + b^4 + (a+b)^4 + W_0{}^4 = T_0{}^n$, then $2(a^2 + ab + b^2)^2 =$

T_0^$n - W_0{}^4$

Hence, (a,b) is parameterized by k as below.

$$a = \frac{Q\ (k)}{P(k)}, b\ = \frac{R\ (k)}{P(k)}, P(k) = 1 + k^2 + Q(k), R(k)\ are\ polinomials\ of\ k$$

$$Q(k)^4 + R(k)^4 + (Q(k) + R(k))^4 + ((1 + k + k^2)\ W_0)^4 = (1 + k + k^2)^{4n}$$

If we find a rational solution of 1 +k +k²=$u^{s/4}$ with s = LCM (n,4), we get the general solution form as below.

$$Q(k)^4\ + R(k)^4\ + (Q(k) + R(k))^4 + (u^{s/4}W_0)^4 = (u^{s/4}T_0)^n$$

7. $X^4 + Y^4 + Z^4 + W^4 = T^5$

$(a, b, c, d)^p = (e)^q$

$(p, q) = (4,5)$

According to $26^4 + 11^4 + 37^4 + 19^4 = 19^5$, we get a parametric solution below.

First, we obtain a parametric solution $a^2 + ab + b^2 = 1083$ using $((a_0, b_0) = (26, 11)$, then we obtain

$$(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W}, \boldsymbol{T})\ =\ (\mathbf{37 - 22K + 26K^2, 11 - 52K - 37K^2,, 26 - 74K - 11K^2, 19 + 19K + 19K^2})$$

Hence, we get

$$X^4 + Y^4 + Z^4 + W^4 = 19^5(1 + k + k^2)^4$$

Set $K = \frac{Q}{P}$. Let $(p - qw) = (m - nw)^5$ where $w = \frac{-1+\sqrt{-3}}{2}$, then we obtaine (p,q) as follows

Next, we find a rational solution of $1 + k + k^2 = u^5$.

Substitute $(p,q) = (m^5 - 10m^3n^2 + n^5 - 10m^2n^3, 5m^4n - 5mn^4 + 10m^3n^2 - n^5)$ to equation (5),

$$X = -37m^{10} - 110nm^9 + 1170n^2m^8 + 4440n^3m^7 + 2310n^4m^6 - 6552n^5m^5 - 7770n^6m^4 - 1320n^7m^3 + 1170n^8m^2 + 370n^9m + 11n^{10},$$

$$Y = 11m^{10} - 260nm^9 - 1665n^2m^8 - 1320n^3m^7 + 5460n^4m^6 + 9324n^5m^5 + 2310n^6m^4 - 3120n^7m^3 - 1665n^8m^2 - 110n^9m + 26n^{10}.$$

then we obtain a parametric solution as follows.

$$Z = -26m^{10} - 370nm^9 - 495n^2m^8 + 3120n^3m^7 + 7770n^4m^6 + 2772n^5m^5 - 5460n^6m^4 - 4440n^7m^3 - 495n^8m^2 + 260n^9m + 37n^{10},$$

$$W = 19(m^2 + nm + n^2)^5,$$

$$T = 19(m^2 + nm + n^2)^4.$$

$(a, b, c, d)^p = (e)^q$

For $(p, q) = (4,5)$

We have, $(11,26,37,19)^4 = (19)^5$

Also, $(6318,2673,8991,4617)^4 = (1539)^5$

We note that in both the above numerical solutions the condition that is met is: $(a + b = c)$

8. $X^4 + Y^4 + Z^4 + W^4 = T^6$

$(a, b, c, d)^p = (e)^q$

$(p, q) = (4,6)$

According to, $(324^4 + 18^4 + 342^4 + 441^4 = 63^6)$

we get a parametric solution below.

In the same way as the case of n = 5, substitute $(p,q) = (m^3 - 3mn^2 - n^3, 3m2n + 3mn^2)$ to equation (5), then we obtain a parametric solution as follows.

$$X = -342m^6 - 108nm^5 + 4860n^2m^4 + 6840n^3m^3 + 270n^4m^2 - 1944n^5m - 342n^6,$$

$$Y = 18m^6 - 1944nm^5 - 5130n^2m^4 - 360n^3m^3 + 4860n^4m^2 + 2052n^5m + 18n^6,$$

$$Z = -54(-2m^2 + 2nm + 3n^2)(m^2 + 6nm + 2n^2)(-3m^2 - 4nm + n^2),$$

$$W = 441(m^2 + nm + n^2)^3,$$

$$T = 63(m^2 + mn + n^2)^2.$$

9. $X^4 + Y^4 + Z^4 + W^4 = T^7$

$(a, b, c, d)^p = (e)^q$

$(p, q) = (4,7)$

According to $65052^4 + 29646^4 + 35406^4 + 75555^4 = 657^7$, we get a parametric solution below.

In the same way as the case of $n = 5$, substitute $(p,q) = (m^7 - 21m^5n^2 + 21m^2n^5 - 35m^4n^3 + 7mn^6$,

$7m6n - 35m^3n^4 + n^7 + 21m^5n^2 - 21m^2n^5)$ to equation (5), then we obtain a parametric-solution as follows.

X= $(35406m^{14} - 415044nm^{13} - 5919732n^2m^{12} - 12887784n^3m^{11} + 29675646n^4m^{10} + 130234104n^5m^9 + 106324218n^6m^8 - 101745072n^7m^7 - 195351156n^8m^6 - 70882812n^9m^5 + 29675646n^{10}m^4 + 23678928n^{11}m^3 + 3221946n^{12}m^2 - 415044n^{13}m - 65052n^{14}$

Y= $(29646m^{14} + 910728nm^{13} + 322194n^2m^{12} - 10791144n^3m^{11} - 65117052n^4m^{10} - 70882812n^5m^5 + 189026938n^7m^7 + 106324218n^9m^5 - 59351292n^9m^5 - 65117052n^{10}m^4 - 12887784n^{11}m^3 + 2697786n^{12}m^2 + 910728n^{13}m + 35406n^{14}$

$$Z = 65052m^{14} + 495684nm^{13} - 2697786n^2m^{12} - 23678928n^3m^{11} - 35441406n^4m^{10} + 59351292n + 195351156n^6m^8 + 121513392n^7m^7 - 89026938n^8m^6 - 130234104n^9m^5 - 35441406n^{10}m^4 + 10791144n^{11}m^3 + 5919732n^{12}m^2 + 495684n^{13}m - 29646n^{14},$$

$$W = 75555(m^2 + nm + n^2)^7$$

$$T = 657(m^2 + nm + n^2)^4$$

10. $X^4 + Y^4 + Z^4 + W^4 = T^9$

$$(a, b, c, d)^p = (e)^q$$

$$(p, q) = (4,9)$$

According to,($647^4 + 601^4 + 46^4 + 361^4 = 19^9$),

we get a parametric solution below.

In the simillar way as the case of n = 5, substitute,

$(p,q) = (m^9 - 36m^7n^2 + 126m^4n^5 - 9mn^8 - 84m^6n^3 + 84m^3n^6 - n^9,\ 9m^8n - 126m^5n^4 + 36m^2n^7 + 36m^7n^2 - 126m^4n^5 + 9mn^8)$ to equation (5),

then we obtain a parametric solution as follows;

X=$(46m^{18} - 10818nm^{17} - 98991n^2m^{16} - 37536n^3m^{15} + 1839060n^4m^{14} + 5543496n^5m^{13} + 853944n^6m^{12} - 19126224n^7m^{11} - 28311426n^8m^{10} - 2236520n^9m^9 + 26298558n^{10}m^8 + 20590128n^{11}m^7 + 853944n^{12}m^6 - 5149368n^{13}m^5 - 1979820n^{14}m^4 - 37536n^{15}m^3 + 91953n^{16}m^2 + 11646n^{17}m + 46n^{18})$

Y= $(601m^{18} + 11646nm^{17} + 7038n^2m^{16} - 490416n^3m^{15} - 1979820n^4m^{14} - 394128n^5m^{13} + 11156964n^6m^{12} + 20590128n^7m^{11} + 2012868n^8m^{10} - 29220620n^9m^9 - 28311426n^{10}m^8 - 1463904n^{11}m^7 + 11156964n^{12}m^6 + 5543496n^{13}m^5 + 140760n^{14}m^4 - 490416n^{15}m^3 - 98991n^{16}m^2 - 828n^{17}m + 601n^{18})$,

Z=$(647m^{18} + 828nm^{17} - 91953n^2m^{16} - 527952n^3m^{15} - 140760n^4m^{14} + 5149368n^5m^{13} + 12010908n^6m^{12} + 1463904n^7m^{11} - 26298558n^8m^{10} - 31457140n^9m^9 - 2012868n^{10}m^8 + 19126224n^{11}m^7 + 12010908n^{12}m^6 +$

$394128n^{13}m^5 - 1839060n^{14}m^4 - 527952n^{15}m^3 - 7038n^{16}m^2 + 10818n^{17}m + 647n^{18})$,

$W = 361(m^2 + nm + n^2)$

$T = 19(m^2 + nm + n^2)^4$

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